uniformly spaced and suitable deflection functions were assumed. The thrust of the paper is to compute the critical buckling load without assuming a deflected shape. The theoretical methods developed here are definitely superior to finite element formulations (Ref. 5) as they are inherently approximate in nature.

To preserve the clarity of the presentation, the twisting and buckling of the edge stiffeners were neglected. However the procedure presented in the paper can be adopted by generating these quantities from the cross sectional properties and the connectivity to the open shell. It is not difficult to adopt the same technique for various prescribed kinematic boundary conditions and/or force boundary conditions.

The symmetry of the system was employed to reduce the general 8×8 matrix to a 4×4 matrix. With the modern day digital computers like CDC 7600, IBM 370 etc. it is not uneconomical to find an iterative solution by using small increments of load instead more logic in programing. The author appreciates the fact that there are other efficient techniques and encourages the use of those methods.

Comment on "Explicit Numerical Method for Solution of Heat-Transfer Problems"

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WITH reference to a recent synoptic by Segletes,[1] the stability criterion quoted by Segletes for the rate-exponential numerical solution of the heat conduction equation is unnecessarily restrictive. To derive the stability criterion directly, in a one-dimensional constant property system with a uniform grid spacing Δx, his Eq. (8) would express

$$T_{i,\theta+\Delta\theta} = \left(T_i e^{-2r} + \tfrac{1}{2}(T_{i+1}+T_{i-1})\ (1-e^{-2r}) + \tfrac{1}{2}(\dot{T}_{i+1}+\dot{T}_{i-1})\Delta\theta[1-(2r)^{-1}(1-e^{-2r})]\right)_\theta \tag{1}$$

where $r=\alpha\Delta\theta/\Delta x^2$, and $\alpha = k/\rho c$. From Eq. (4) of Segletes paper $\dot{T}_{i+1}$ and $\dot{T}_{i-1}$ can be found in terms of T_{i+2}, T_{i+1}, T_i and T_i, T_{i-1}, T_{i-2}, respectively, so that Eq. (1) becomes

$$T_{i,\theta+\Delta\theta} = (T_i e^{-2r} + \tfrac{1}{2}(T_{i+1}+T_{i-1})(1-e^{-2r}) + r/2\ (T_{i+2}-2T_{i+1}+2T_i-2T_{i-1}+T_{I-2}) \cdot [1-(2r)^{-1}(1-e^{-2r})])_\theta \tag{2}$$

For stability, the coefficients of the temperature terms on the right hand side of this equation must sum to unity and must all be positive. This condition is satisfied provided that

$$1-e^{-2r}-r \geq 0 \tag{3}$$

The previous expression is the coefficient of $T_{i\pm 1}$, all other coefficients are positive for all positive values of r. The maximum value of r for which inequality (3) is satisfied is

$$r = 0.7968 \tag{4}$$

The well know result for the conventional explicit finite difference method is $r=0.5$, so Segletes' method allows for a 59.4% enhancement of the maximum allowable time step in this case, much better than the 15% cited by him.

Secondly, as Fig. 2 of Segletes' paper shows, both his method and the conventional method are superior to Larkin's method, Eq. (1) with $\dot{T}_{i+1}=\dot{T}_{i-1}=0$. This is due to the inconsistency of Larkin's method with the governing differential equation. By differencing old and new temperatures in the same heat flux expressions, Larkin's method leads to 0 $(\Delta\theta/\Delta x^2)$ terms in the resulting truncation error in the present one-dimensional example. To provide time increments that are competitive with those permitted by the Segletes' and the conventional methods, $\Delta\theta$ should be at least $0(\Delta x^2)$ for the Larkin method. If this is the case, the $0(\Delta\theta/\Delta x^2)$ term will not vanish from the truncation error as $\Delta x \to 0$ and consequently Larkin's procedure will not provide a solution of the heat equation

$$\frac{\partial T}{\partial\theta} = \alpha\frac{\partial^2 T}{\partial x^2} \tag{5}$$

but it will be a solution of

$$\frac{\partial T}{\partial\theta}\left(1+2r\frac{\theta}{\Delta\theta}\right) = \alpha\frac{\partial^2 T}{\partial x^2} \tag{6}$$

where $\theta=0$ at the start of a time step and $\theta=\Delta\theta$ at the end of the time step. After one time step, one can treat an interior grid point as if it were embedded in an infinite solid. Under this assumption, the solution of Eq. (5) after one time step is

$$T(x,\Delta\theta) = \int_{-\infty}^{+\infty} \frac{T(x',0)}{2(\pi\alpha\Delta\theta)^{1/2}} \exp\left\{\frac{-(x-x')^2}{4\alpha\Delta\theta}\right\} dx' \tag{7}$$

whereas the corresponding solution of Eq. (6) is

$$T(x,\Delta\theta) = \int_{-\infty}^{+\infty} \frac{T(x',0)}{2\left[\frac{\pi\alpha\Delta\theta}{2r}\ln(1+2r)\right]^{1/2}} \exp\left\{\frac{-(x-x')^2}{\frac{4\alpha\Delta\theta}{2r}\ln(1+2r)}\right\} dx' \tag{8}$$

Comparison of these 2 equations shows that the Larkin method provides a numerical solution to the heat equation with an equivalent thermal diffusivity α_e where

$$\alpha_e = (\alpha/2r)\ \ln(1+2r) \tag{9}$$

According to Carslaw and Jaeger,[2] the terminal slopes of the temperature vs time curves in Fig. 2 of Segletes' paper are proportional to α. The correct terminal slope is about 72° F/sec., a value with which Segletes' and the conventional methods agree. Table 1 shows the value of the time steps considered by Segletes and the corresponding Larkin terminal slope values found using Eq. (9).

The slopes listed agree well with the Larkin terminal slopes in Segletes' Fig. 2. The time steps listed all correspond to the

Table 1 Time steps and Larkin slopes

$\Delta\theta$(sec)	Larkin Slope (°F/sec)
0.100	46.3
0.050	55.8
0.005	69.8

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same value of Δx, so that r is proportional to $\Delta\theta$. From Eq. (9) $\alpha_e \mapsto \alpha$ as $r \mapsto 0$, which is why the Larkin curves approach the correct curves in Segletes' Fig. 2 as $\Delta\theta$ decreases. The apparent objective of Larkin's method is to allow large values of r to be used to obtain good temperature histories; the goal is obviously defeated by the method's inconsistency. By introducing Eq. (5) in his paper, Segletes has derived a consistent method. The conventional method is also consistent. It is, therefore, no surprise that Segletes' and the conventional methods produce better results than Larkin's method, for as Lax and Richtmeyer[3] showed, both stability *and* consistency are necessary requirements for the convergence of a numerical solution of a partial differential equation to its exact solution as the relevant grid spacings are shrunk.

References

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[2] Carslaw, H.S. and Jaeger, J.C., *Conduction of Heat is Solids*, 2nd Ed. Oxford University Press, London, 1959, p. 112.

[3] Lax, P.D. and Richtmeyer, R.D., "Survey of the Stability of Linear Finite Difference Equations," *Communications on Pure and Applied Mathematics*, Vol. 9, May 1956, pp. 267-293.